COMPUTATION OF THE DAMPING FACTOR FOR FREE OSCILLATIONS OF A VISCOUS LIQUID IN A CYLINDRICAL VESSEL

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 143-146, 1965

The damping factor for free oscillations of a viscous liquid in a circular cylindrical vessel is determined by calculating the energy dissipated during a period. The dissipated energy is determined separately in and outside the boundary layer, the motion of the liquid outside this layer being assumed to be identical with the motion of a perfect liquid.

1. Consider the standing oscillations of a viscous heavy liquid of depth h in a cylindrical vessel of radius a. We assume that everywhere outside the boundary layer the liquid moves like a perfect liquid, and in determining this motion we neglect the thickness of the boundary layer.

Then the velocity potential for the S-th mode of free oscillation will be [1]

$$\varphi_s = \frac{2a^2\zeta_s\omega_s}{\operatorname{ch}(\zeta_s h / a)(\zeta_s^2 - 1) J_1(\zeta_s)} J_1\left(\zeta_s \frac{r}{a}\right) \operatorname{ch}\left(\zeta_s \frac{z+h}{a}\right) \sin\eta \sin\omega_s t . \tag{1.1}$$

Here, r, η, z are cylindrical coordinates with origin at the center of the free surface of the liquid, t is time, J_1 a first-order Bessel function of the first kind, ζ_s the positive roots of the frequency equation $J_1'(\zeta) = 0$, ω_s the frequency of the S-th mode of oscillation of the liquid, and g the acceleration of gravity.

The frequency ω_s is given by

$$\omega_s^2 = \zeta_s \frac{g}{a} \operatorname{th}\left(\zeta_s \frac{h}{a}\right).$$

2. The term $\sin \omega_s t$ in (1.1) was obtained upon separation of variables. We assume that the damping can be approximately taken into account by formally introducing a factor $\exp(-n_s t)$ in (1.1), i.e., we assume that at any moment, even in the presence of friction, the liquid moves like a perfect liquid (everywhere except in the boundary layer). Comparison of the maximum values of the kinetic energy of the liquid in the S-th mode of oscillation leads to the expression

$$n_s = \frac{\omega_s}{4\pi} \ln \frac{T_s(t_1)}{T_s(t_1+T)}. \tag{2.1}$$

Here T_s is the kinetic energy of the liquid in the S-th mode, and T is the period.

3. The dissipation of energy per unit time is given by the integral

$$A = \mu \int_{(V)} (2\varepsilon_1^2 + 2\varepsilon_2^2 + 2\varepsilon_3^2 + \theta_1^2 + \theta_2^2 + \theta_3^2)\, dV, \tag{3.1}$$

Here is the viscosity of the liquid, ε_1, ε_2, ε_3, $1/_2\theta_1$, $1/_2\theta_2$, $1/_2\theta_3$ are the components of the strain rate tensor related as follows with the projections v_r, v_η, v_z of the liquid velocities onto the axes of the cylindrical coordinate system:

$$\varepsilon_1 = \frac{\partial v_r}{\partial r}, \qquad \varepsilon_2 = \frac{1}{r}\left(\frac{\partial v_\eta}{\partial \eta} + v_r\right), \qquad \varepsilon_3 = \frac{\partial v_z}{\partial z},$$
$$\theta_1 = \frac{\partial v_\eta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \eta}, \qquad \theta_2 = \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}, \qquad \theta_3 = \frac{1}{r}\frac{\partial v_r}{\partial \eta} + \frac{\partial v_\eta}{\partial r} - \frac{v_\eta}{r}. \tag{3.2}$$

Having determined the velocities v_r, v_η, v_z from the expression for the potential (1.1), we evaluate the integral (3.1) over the entire volume of the liquid. Assuming that h is sufficiently large, so that

$$2h \ll \frac{a}{\zeta_s} \operatorname{sh}\left(2\frac{\zeta_s h}{a}\right), \qquad \operatorname{ch}^2\left(\zeta_s \frac{h}{a}\right) \approx \frac{1}{2} \operatorname{sh}\left(2\zeta_s \frac{h}{a}\right),$$

after integration and a series of transformations we get

$$A_s = 4\pi\mu a^3 \omega_s{}^2 \Phi(\zeta_s) \sin^2 \omega_s t, \tag{3.3}$$

where

$$\Phi(\zeta_s) = \frac{\zeta_s}{(\zeta_s^2 - 1)^2 J_1^2(\zeta_s)} \left[2(\zeta_s^2 - 1) J_1^2(\zeta_s) + 1 - J_1^2(\zeta_s)\left(1 + \frac{2}{\zeta_s^2}\right) - 2\int_0^{\zeta_s} \frac{1}{x} J_1^2(x)\, dx\right]. \tag{3.4}$$

Then the dissipation of energy during a period will be

$$P_s = \int_0^\tau A_s\, dt = 4\pi\mu a^3 \omega_s \Phi(\zeta_s) \left(\tau = \frac{2\pi}{\omega_s}\right). \tag{3.5}$$

4. We can simplify the computation of the energy losses in the boundary layer by noting that at the side walls of the cylinder the terms with $\partial v_z/\partial r$ and $\partial v_\eta/\partial r$ considerably exceed the other components of the integrand in (3.1). We discard the above-mentioned small terms and disregard the boundary layer at the bottom of the cylinder. In order to find the terms with $\partial v_z/\partial r$ and $\partial v_\eta/\partial r$ we solve two auxiliary problems.

In the first of these, an infinite cylinder of radius a, filled with a viscous incompressible liquid rotates about its axis, so that its walls have a velocity $v = v_0 \sin \omega_s t$. It is required to find the velocity distribution in the liquid for steady motion.

For the first problem the Navier-Stokes equations degenerate into the single equation

$$\frac{\partial v_\eta}{\partial t} = \nu \left(\frac{\partial^2 v_\eta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\eta}{\partial r} - \frac{v_\eta}{r^2} \right), \quad v_r = v_z = 0,$$

with boundary condition

$$v_\eta(a, t) = v_0 \sin \omega_s t.$$

Here ν is the kinematic viscosity. Finding a solution in the form

$$v_\eta(r, t) = R(r)\, T(t),$$

we have

$$v_\eta(r, t) = v_0 \psi_1(r, t) = \frac{v_0}{\mathrm{ber}_1^2(a\rho_s) + \mathrm{bei}_1^2(a\rho_s)} \Big\{ \mathrm{bei}_1(a\rho_s)\, \mathrm{ber}_1(r\rho_s) - \mathrm{ber}_1(a\rho_s)\, \mathrm{bei}_1(r\rho_s)] \cos \omega_s t + [\mathrm{bei}_1(a\rho_s)\, \mathrm{bei}_1(r\rho_s) + \mathrm{ber}_1(a\rho_s)\, \mathrm{ber}_1(r\rho_s)] \sin \omega t \Big\} \tag{4.1}$$

$$(\rho_s = \sqrt{\omega_s/\nu}).$$

The functions $\mathrm{ber}_1(x)$ and $\mathrm{bei}_1(x)$ are given by

$$\mathrm{ber}_1(x) + i\, \mathrm{bei}_1(x) = -J_1(x^{-1/4 i\pi}).$$

For the second problem the Navier-Stokes equations also degenerate into a single equation

$$\frac{\partial v_z}{\partial t} = \nu \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} \right), \qquad v_r = v_\eta = 0,$$

with boundary condition

$$v_z(a, t) = v_0 \sin \omega_s t.$$

Solving this problem, for the velocity we obtain

$$v_z(r, t) = v_0 \psi_0(r, t) = \frac{v_0}{\mathrm{ber}_0^2(a\rho_s) + \mathrm{bei}_0^2(a\rho_s)]} \{ [\mathrm{bei}_0(a\rho_s)\, \mathrm{ber}_0(a\rho_s) - \mathrm{ber}_0(a\rho_s)\, \mathrm{bei}_0(r\rho_s)] \cos \omega_s t + [\mathrm{bei}_0(a\rho_s)\, \mathrm{bei}_0(r\rho_s) + \mathrm{ber}_0(a\rho_s)\, \mathrm{ber}_0(r\rho_s)] \sin \omega_s t \} \tag{4.2}$$

$$(\rho_s = \sqrt{\omega_s/\nu}).$$

Here, as before,

$$\mathrm{ber}_0(x) + i\, \mathrm{bei}_0(x) = J_0(x e^{-1/4 \pi i}).$$

We assume that the liquid velocity at the interface between the boundary layer and the inner region is equal to the velocity of a perfect liquid at the walls of the vessel. This enables us to find the velocity distribution in the boundary layer

$$v_z(r, \eta, z, t) = \left[\frac{1}{\sin \omega s t} \frac{\partial \varphi_s}{\partial z} \right]_{r=a} (\sin \omega_s t - \psi_0(r, t)), \tag{4.3}$$

$$v_\eta(r, \eta, z, t) = \left[\frac{1}{\sin \omega_s t} \frac{\partial \varphi_s}{a\, \partial \eta} \right]_{r=a} (\sin \omega_s t - \psi_1(r, t)). \tag{4.4}$$

If it is noted that for liquids of low viscosity $a\rho_s \gg 1$ (for water, for example, when $a = 5$ cm and $\nu = 0.01$ cm²/sec we have $a\rho_1 = 215$), then, using the known asymptotic expansions of the functions $\mathrm{ber}_n(x)$ and $\mathrm{bei}_n(x)$ we can reduce the expressions (4.3) and (4.4) to the simpler form

$$v_z = \frac{2a\zeta_s\omega_s}{\mathrm{ch}(\zeta_s h/a)(\zeta_s^2-1)} \mathrm{sh}\left(\zeta_s \frac{z+h}{a}\right) \sin\eta \Big\{\sin\omega_s t - \\ - \frac{\sqrt{a}}{\sqrt{r}} \exp\left[-(a-r)\frac{\rho_s}{\sqrt{2}}\right] \sin\left[(a-r)\frac{\rho_s}{\sqrt{2}} + \omega_s t\right]\Big\}, \tag{4.5}$$

$$v_\eta = \frac{2a\omega_s}{\mathrm{ch}(\zeta_s h/a)(\zeta_s^2-1)} \mathrm{ch}\left(\zeta_s \frac{z+h}{a}\right) \cos\eta \Big\{\sin\omega_s t - \\ - \frac{\sqrt{a}}{\sqrt{r}} \exp\left[-(a-r)\frac{\rho_s}{\sqrt{2}}\right] \sin\left[(a-r)\frac{\rho_s}{\sqrt{2}} + \omega_s t\right]\Big\}. \tag{4.6}$$

Evaluating integral (3.1) over the volume and then integrating with respect to time, we get the energy losses during a period in the boundary layer:

$$P_s' = \frac{\sqrt{2}\pi^2\mu a^4\rho_s\omega_s(\zeta_s^2+1)}{\zeta_s(\zeta_s^2-1)^2}. \tag{4.7}$$

5. In order to determine the damping factor it is also necessary to know the total energy of the liquid

$$T_s = \frac{d}{2}\omega_s^2 \int_{(V)} \left\{\mathrm{grad}\left[\frac{\varphi_s}{\sin\omega_s t}\right]\right\}^2 dV. \tag{5.1}$$

Here d is the density of the liquid. Evaluating integral (5.1), we get

$$T_s = \frac{\pi d a^5\omega_s^2}{\zeta_s(\zeta_s^2-1)} \mathrm{th}\left(\zeta_s\frac{h}{a}\right). \tag{5.2}$$

Since $T_s(t_1+T) = T_s(t_1) - P_s - P_s'$, in accordance with (2.1),

$$n_s = \frac{\omega_s}{4\pi} \ln\frac{T_s}{T_s - P_s - P_s'}. \tag{5.3}$$

But $P_s + P_s' \ll T_s$; therefore (5.3) may be approximately rewritten in the form

$$n_s = \frac{\omega_s}{4\pi}\frac{P_s+P_s'}{T_s}. \tag{5.4}$$

In (3.4) the first term in square brackets is greater than the others and the difference between terms rapidly increases in s. This permits the simplification of (5.4). Taking into account only the first term in (3.4), putting $\mathrm{th}(\zeta_s h/a) \approx 1$, and substituting (3.5), (4.7) and (5.2) into (5.4), we get

$$n_s = \frac{2\nu\zeta_s^2}{a^2} + \frac{\nu\rho_s}{2\sqrt{2}a}\frac{\zeta_s^2+1}{\zeta_s^2-1}. \tag{5.5}$$

In this expression the first term characterizes the energy losses outside the boundary layer, the second those in the boundary layer. It is easy to see that as far as the first modes of oscillation are concerned the main role is played by the energy losses in the boundary layer.

6. The above method of calculating the damping factor applies to liquids that do not form films of adsorbed matter on the free surface. The presence of such a film leads to the appearance of a boundary layer in its vicinity and to a sharp increase in the damping factor [2]. The effect of a film can be roughly taken into account by assuming that it cannot be deformed in its plane and calculating the additional energy losses as for a plane boundary layer.

G. N. Mikishev and N. Ya. Dorozhkin [3] recently published the results of an experimental investigation of the free oscillations of various liquids, and these may be used to check the proposed method of determining the damping factor. In comparing the calculations with the experimental data it is necessary to bear in mind that in [3] the quantity nT was used as the logarithmic decrement, rather than the usual 0.5nT.

A comparison of the experimental and calculated values for liquids not forming surface films shows quite good agreement. Thus, at a cylinder radius $a = 10$ cm in the case of acetone ($\nu = 0.6$) the calculated value of the damping factor for oscillations of the first mode is 21% below the experimental figure, while in the case of turpentine oil ($\nu = 1.80$) the corresponding value is 12% below. In the case of water with a small admixture of glycerin the experimental value for the damping factor is three times greater than the calculated value owing to neglect of the surface film effect.

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10 October 1962 Leningrad